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The metrizability of L -topological groups

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Abstract In this study, we study the metrizability of the notion of L -topological group defined by Ahsanullah 1988. We show that for any (separated) L -topological group there is an L -pseudo-metric (L -metric), in sense of Gähler which is defined using his notion of L -real numbers, compatible with the L -topology of this (separated) L -topological group. That is, any (separated) L -topological group is pseudo-metrizable (metrizable).

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1. Introduction

The notion of L -real numbers is defined and studied by S. Gähler and W. Gähler in [1]. $\mathbf{R}_L$ denotes the set of all L -real numbers. The subset $\mathbf{R}_L^*$ of $\mathbf{R}_L$ of all positive L -real numbers is used to define the L -pseudo-metric (L -metric) on a set X , by the same authors in [1], as a mapping of the cartesian product $X \times X$ to $\mathbf{R}_L^*$ which satisfying similar conditions to the conditions of the usual metric. In this paper we study the metrizability, using the L -pseudo-metric (L -metric) in sense of [1], of a notion of L -topological group which is introduced

in [2] and studied in [3]. This L -topological groups is defined as a group equipped with an L -topology such that both the binary operation and the unary operation of the inverse are L -continuous with respect to this L -topology.

In this paper, using the uniformizability of L -topological groups introduced by the authors in [4], we show that any (separated) L -topological group is pseudo-metrizable (metrizable). In [4] is used the L -uniform structures which are defined in [5] on a set X , in a similar way to the usual case, as L -filters on $X \times X$.

In Section 2 of this paper we recall some results on L -filters, L -real numbers defined by Gähler in [1,6–8], and some separation axioms defined by the authors in [9–12].

Sections 3 and 4 introduce and show some results on L -metric and L -uniform spaces, respectively, which are needed to show the metrizability of L -topological groups. We will use the notion of L -topogenous structure [13].

In Section 5 we show that the L -pseudo-metric (L -metric), in sense of [1], induces the L -topology of a (separated) L -topological group, that is, any (separated) L -topological group is pseudo-metrizable (metrizable).

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2. On L -filters

Recall here some ideas concerning L -filters needed in the paper. Denote by L^X the set of all L -subsets of a non-empty set X , where L is a complete chain with different least and greatest elements 0 and 1, respectively [14]. Let $L_0 = L \setminus \{0\}$ and $L_1 = L \setminus \{1\}$. For each L -set $\lambda \in L^X$, let λ' denote the complement of λ , defined by $\lambda'(x) = \lambda(x)'$ for all $x \in X$. For all $x \in X$ and $\alpha \in L_0$, the L -subset x_α of X whose value α at x and 0 otherwise is called an L -point in X and the constant L -subset of X with value α will be denoted by $\bar{\alpha}$.

L -filters. By an L -filter on a non-empty set X we mean [7] a mapping $\mathcal{M} : L^X \rightarrow L$ such that $\mathcal{M}(\bar{\alpha}) \leq \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\bar{1}) = 1$, and also $\mathcal{M}(\lambda \wedge \mu) = \mathcal{M}(\lambda) \wedge \mathcal{M}(\mu)$ for all $\lambda, \mu \in L^X$. $\mathcal{M}$ is called *homogeneous* [7] if $\mathcal{M}(\bar{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are L -filters on X , $\mathcal{M}$ is called *finer than* $\mathcal{N}$, denoted by $\mathcal{M} \succ \mathcal{N}$, provided $\mathcal{M}(\lambda) \geq \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$. By $\mathcal{M} \not\succ \mathcal{N}$ we mean that $\mathcal{M}$ is not finer than $\mathcal{N}$. Since L is a complete chain, then

$$\mathcal{M} \not\succ \mathcal{N} \iff \text{there is } f \in L^X \text{ such that } \mathcal{M}(f) < \mathcal{N}(f).$$

Let $\mathcal{F}_L X$ denote the set of all L -filters on X , $f : X \rightarrow Y$ a mapping and $\mathcal{M}, \mathcal{N}$ are L -filters on X, Y , respectively. Then the *image* of $\mathcal{M}$ and the *preimage* of $\mathcal{N}$ with respect to f are the L -filters $\mathcal{F}_L f(\mathcal{M})$ on Y and $\mathcal{F}_L^{-1} f(\mathcal{N})$ on X defined by $\mathcal{F}_L f(\mathcal{M})(\mu) = \mathcal{M}(\mu \circ f)$ for all $\mu \in L^Y$ and $\mathcal{F}_L^{-1} f(\mathcal{N})(\lambda) = \bigvee_{\mu \circ f \leq \lambda} \mathcal{N}(\mu)$ for all $\lambda \in L^X$, respectively. For each mapping $f : X \rightarrow Y$ and each L -filter $\mathcal{N}$ on Y , for which the preimage $\mathcal{F}_L^{-1} f(\mathcal{N})$ exists, we have $\mathcal{F}_L f(\mathcal{F}_L^{-1} f(\mathcal{N})) \succ \mathcal{N}$. Moreover, for each L -filter $\mathcal{M}$ on X , the inequality $\mathcal{M} \succ \mathcal{F}_L^{-1} f(\mathcal{F}_L f(\mathcal{M}))$ holds [7].

For any set A of L -filters on X , the infimum $\bigwedge_{\mathcal{M} \in A} \mathcal{M}$, with respect to the finer relation on L -filters, does not exist in general. The infimum $\bigwedge_{\mathcal{M} \in A} \mathcal{M}$ of A exists *if and only if* for each non-empty finite subset $\{\mathcal{M}_1, \dots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(\lambda_1) \wedge \dots \wedge \mathcal{M}_n(\lambda_n) \leq \sup(\lambda_1 \wedge \dots \wedge \lambda_n)$ for all $\lambda_1, \dots, \lambda_n \in L^X$ [6]. If the infimum of A exists, then for each $\lambda \in L^X$ and n as a positive integer we have

$$\left(\bigwedge_{\mathcal{M} \in A} \mathcal{M} \right)(\lambda) = \bigvee_{\substack{\lambda_1 \wedge \dots \wedge \lambda_n \leq \lambda \\ \mathcal{M}_1, \dots, \mathcal{M}_n \in A}} (\mathcal{M}_1(\lambda_1) \wedge \dots \wedge \mathcal{M}_n(\lambda_n)).$$

By a *filter* on X we mean a non-empty subset $\mathcal{F}$ of L^X which does not contain $\bar{0}$ and closed under finite infima and super sets [15]. For each L -filter $\mathcal{M}$ on X , the subset α -pr $\mathcal{M}$ of L^X defined by: α -pr $\mathcal{M} = \{\lambda \in L^X \mid \mathcal{M}(\lambda) \geq \alpha\}$ is a filter on X .

A family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of non-empty subsets of L^X is called *valued L -filter base* on X [7] if the following conditions are fulfilled:

$$(V1) \quad \lambda \in \mathcal{B}_\alpha \text{ implies } \alpha \leq \sup \lambda.$$

$$(V2) \quad \text{For all } \alpha, \beta \in L_0 \text{ and all } L\text{-sets } \lambda \in \mathcal{B}_\alpha \text{ and } \mu \in \mathcal{B}_\beta, \text{ if even } \alpha \wedge \beta > 0 \text{ holds, then there are a } \gamma \geq \alpha \wedge \beta \text{ and an } L\text{-set } v \leq \lambda \wedge \mu \text{ such that } v \in \mathcal{B}_\gamma.$$

Each valued L -filter base $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ on a set X defines an L -filter $\mathcal{M}$ on X by: $\mathcal{M}(\lambda) = \bigvee_{\mu \in \mathcal{B}_\alpha, \mu \leq \lambda} \alpha$ for all $\lambda \in L^X$. On the other hand, each L -filter $\mathcal{M}$ can be generated by many valued L -filter bases, and among them the greatest one $(\alpha\text{-pr } \mathcal{M})_{\alpha \in L_0}$.

L -neighborhood filters. In the following, the topology in sense of [16,17] will be used which will be called L -topology. int_τ and cl_τ denote the interior and the closure operators with respect to the L -topology τ , respectively. For each L -topological space (X, τ) and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \rightarrow L$ defined by: $\mathcal{N}(x)(\lambda) = \text{int}_\tau \lambda(x)$ for all $\lambda \in L^X$ is an L -filter on X , called the *L -neighborhood filter* of the space (X, τ) at x , and the mapping $\dot{x} : L^X \rightarrow L$ defined by $\dot{x}(\lambda) = \lambda(x)$ for all $\lambda \in L^X$ is a homogeneous L -filter on X . The L -neighborhood filters fulfill the following conditions:

$$(N1) \quad \dot{x} \succ \mathcal{N}(x) \text{ holds for all } x \in X;$$

$$(N2) \quad (\mathcal{N}(x))(\text{int}_\tau f) = (\mathcal{N}(x))(f) \text{ for all } x \in X \text{ and } f \in L^X.$$

Let (X, τ) and (Y, σ) be two L -topological spaces. Then the mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is called *L -continuous* (or (τ, σ) -continuous) provided $\text{int}_\sigma \mu \circ f \leq \text{int}_\tau (\mu \circ f)$ for all $\mu \in L^Y$ [8].

The L -neighborhood filter $\mathcal{N}(F)$ at an ordinary subset F of X is the L -filter on X defined, by the authors in [10], by means of $\mathcal{N}(x)$, $x \in F$ as: $\mathcal{N}(F) = \bigvee_{x \in F} \mathcal{N}(x)$. The L -filter $\dot{F}$ is defined by: $\dot{F} = \bigvee_{x \in F} \dot{x}$. $\dot{F} \succ \mathcal{N}(F)$ holds for all subsets F of X . Recall also here the L -filter $\dot{\lambda}$ and the L -neighborhood filter $\mathcal{N}(\lambda)$ at an L -subset λ of X which are defined by

$$\dot{\lambda} = \bigvee_{0 < \lambda(x)} \dot{x} \quad \text{and} \quad \mathcal{N}(\lambda) = \bigvee_{0 < \lambda(x)} \mathcal{N}(x), \quad (2.1)$$

respectively. $\dot{\lambda} \succ \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$ [18].

L -real numbers. By an L -real number is meant [1] a convex, normal, compactly supported and upper semi-continuous L -subset of the set of all real numbers $\mathbf{R}$. The set of all L -real numbers is denoted by $\mathbf{R}_L$. $\mathbf{R}$ is canonically embedded into $\mathbf{R}_L$, identifying each real number a with the crisp L -number $a^\sim$ defined by $a^\sim(\xi) = 1$ if $\xi = a$ and 0 otherwise. The set of all positive L -real numbers is defined and denoted by: $\mathbf{R}_L^* = \{x \in \mathbf{R}_L \mid x(0) = 1 \text{ and } 0^\sim \leq x\}$ and let $I_L = \{x \in \mathbf{R}_L^* \mid x \leq 1^\sim\}$, where $I = [0, 1]$ is the real unit interval. Note that we mean here by $\leq$ the binary operation on $\mathbf{R}_L$ defined by

$$x \leq y \iff x_{\alpha_1} \leq y_{\alpha_1} \quad \text{and} \quad x_{\alpha_2} \leq y_{\alpha_2}$$

for all $x, y \in \mathbf{R}_L$ where $x_{\alpha_1} = \inf\{z \in \mathbf{R} \mid x(z) \geq \alpha\}$ and $x_{\alpha_2} = \sup\{z \in \mathbf{R} \mid x(z) \geq \alpha\}$ for all $x \in \mathbf{R}_L$ and for all $\alpha \in L_0$. It is shown in [7] that the class $\{R_\delta|_{I_L} \mid \delta \in I\} \cup \{R^\delta|_{I_L} \mid \delta \in I\} \cup \{0^\sim|_{I_L}\}$ is a base for an L -topology $\mathfrak{I}$ on I_L , where R^δ and R_δ are the L -subsets of $\mathbf{R}_L$ defined by $R_\delta(x) = \bigvee_{\alpha > \delta} x(\alpha)$ and $R^\delta(x) = (\bigvee_{\alpha \geq \delta} x(\alpha))'$ for all $x \in \mathbf{R}_L$ and $\delta \in \mathbf{R}$ and note that $R_\delta|_{I_L}, R^\delta|_{I_L}$ are the restrictions of R_δ, R^δ on I_L , respectively. Recall also that $x \pm y$ are L -real numbers defined by $(x \pm y)(\xi) = \bigvee_{\eta, \zeta \in \mathbf{R}, \eta \pm \zeta = \xi} (x(\eta) \wedge y(\zeta))$ for all $\xi \in \mathbf{R}$. $(\mathbf{R}_L, +)$ is a commutative semi group with identity element $0^\sim$. The positive part x^+ of an L -real number x is defined as $x^+ = 0^\sim \vee x$, where

$$x - x = 0^\sim, \quad (x + y)^+ \leq x^+ + y^+. \quad (2.2)$$

GT_τ -spaces. An L -topological space (X, τ) is called [9,11]:

- (1) GT_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\succ \mathcal{N}(y)$ or $\dot{y} \not\succ \mathcal{N}(x)$.
- (2) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\succ \mathcal{N}(y)$ and $\dot{y} \not\succ \mathcal{N}(x)$.

- (3) *completely regular* if for all $x \notin F$ and $F = \text{cl}_\tau F$, there exists an L -continuous mapping $f : (X, \tau) \rightarrow (I_L, \mathfrak{I})$ such that $f(x) = \bar{1}$ and $f(y) = \bar{0}$ for all $y \in F$.
- (4) $GT_{\frac{3}{2}}$ (or L -Tychonoff) if it is GT_1 and completely regular.

Proposition 2.1 (9–12). *Every GT_i -space is GT_{i-1} -space for all $i = 1, 2, 3, 4, 5, 6$. Moreover, the implications between GT_3 -spaces, $GT_{\frac{3}{2}}$ -spaces and GT_4 -spaces goes as expected.*

3. Some results on L -metric spaces

A mapping $q : X \times X \rightarrow \mathbf{R}_L^*$ is called an L -metric [1] on X if the following conditions are fulfilled:

- (1) $q(x, y) = 0^\sim$ if and only if $x = y$
 (2) $q(x, y) = q(y, x)$
 (3) $q(x, y) \leq q(x, z) + q(z, y)$.

If $q : X \times X \rightarrow \mathbf{R}_L^*$ satisfies the conditions (2) and (3) and the following condition:

- (1)' $q(x, y) = 0^\sim$ if $x = y$

then it is called an L -pseudo-metric on X .

A set X equipped with an L -pseudo-metric (L -metric) q on X is called an L -pseudo-metric (L -metric) space.

To each L -pseudo-metric (L -metric) q on a set X is generated canonically a stratified L -topology τ_q on X which has $\{\varepsilon \circ q_x | \varepsilon \in \mathcal{E}, x \in X\}$ as a base, where $q_x : X \rightarrow \mathbf{R}_L^*$ is the mapping defined by $q_x(y) = q(x, y)$ and

$$\mathcal{E} = \{\bar{\alpha} \wedge R^\delta |_{\mathbf{R}_L^*} | \delta > 0, \alpha \in L\} \cup \{\bar{\alpha} | \alpha \in L\},$$

here $\bar{\alpha}$ has $\mathbf{R}_L^*$ as domain.

An L -topological space (X, τ) is called *pseudo-metrizable* (*metrizable*) if there is an L -pseudo-metric (L -metric) q on X inducing τ , that is, $\tau = \tau_q$.

An L -pseudo-metric q is called *left (right) invariant* if

$$q(x, y) = q(ax, ay) (q(x, y) = q(xa, ya)) \text{ for all } a, x, y \in X.$$

An L -set $\lambda \in L^X$ is called *countable (finite)* if its support is countable (finite), where the support of λ is the set $\{x \in X | 0 < \lambda(x)\}$.

Let us call an L -filter $\mathcal{M}$ on a set X *countable* if the sets $\alpha - \text{pr} \mathcal{M}$ are countable for all $\alpha \in L_0$.

Definition 3.1. An L -topological space (X, τ) is called *first countable* if every point $x \in X$ has a countable L -neighborhood filter $\mathcal{N}(x)$.

Proposition 3.1. *For any L -pseudo-metric q on a set X , if τ_q is the L -topology associated with q , then (X, τ_q) is a first countable space.*

Proof. Since $\{\varepsilon \circ q_x | \varepsilon \in \mathcal{E}, x \in X\}$ is a base for τ_q , then for all $n \in \mathbf{N}$, the set $B_n = \{\varepsilon_n \circ q_x | \varepsilon_n \in \mathcal{E}, x \in X\}$, where $\varepsilon_n = \frac{1}{n} \wedge R^\delta |_{\mathbf{R}_L^*}$, is the $\frac{1}{n} - \text{pr} \mathcal{N}(x)$, which implies that there exists a countable L -neighborhood filter $\mathcal{N}(x)$ at every point $x \in X$. Hence, (X, τ_q) is a first countable space. $\square$

By an L -function family Φ on a set X , we mean the set of all L -real functions $f : X \rightarrow I_L$.

We also have the following results.

Lemma 3.1. *Let Φ be an L -function family on X and $\sigma_f : X \times X \rightarrow I_L$ a mapping defined by*

$$\sigma_f(x, y) = (f(x) - f(y))^+, \quad f \in \Phi.$$

Then σ_f is an L -pseudo-metric on X .

Proof. Clearly, $\sigma_f(x, y) = \sigma_f(y, x)$. From (2.1), we get that $\sigma_f(x, x) = (f(x) - f(x))^+ = 0^\sim$ for all $x \in X$, and moreover

$$\begin{aligned} \sigma_f(x, y) &= (f(x) - f(y))^+ \leq (f(x) - f(z))^+ + (f(z) - f(y))^+ \\ &= \sigma_f(x, z) + \sigma_f(z, y). \end{aligned}$$

Hence, σ_f is an L -pseudo-metric on X . $\square$

Lemma 3.2. *Let $\sigma_i : X \times X \rightarrow I_L, i \in I$ be an arbitrary set of L -pseudo-metrics on the set X . Then*

$$\sigma(x, y) = \sup\{\sigma_i(x, y) | i \in I\}$$

defines an L -pseudo-metric on X as well.

Proof. Only the triangle inequality has to be shown. For all $x, y, z \in X$ and all $i \in I$, we have

$$\sigma_i(x, y) \leq \sigma_i(x, z) + \sigma_i(z, y) \leq \sigma(x, z) + \sigma(z, y),$$

and then $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$. Hence, σ is an L -pseudo-metric on X . $\square$

Here, we have shown this fact.

Lemma 3.3. *Any L -pseudo-metric q on a set X is an L -metric on X if and only if (X, τ_q) is a GT_0 -space.*

Proof. Let $x, y \in X$ and $y \neq x$. Since (X, τ_q) is a GT_0 -space, then there exists $\mu \in L^X$ such that $\mu(x) < \beta \leq \text{int}_{\tau_q} \mu(y)$ for some $\beta \in L_0$. From the definition of the base of τ_q , since

$$\text{int}_{\tau_q} \mu(z) = \bar{\alpha} \wedge R^\delta |_{\mathbf{R}_L^*} (q(x, z)) = \alpha \wedge \left(\bigvee_{\eta \geq \delta} q(x, z)(\eta) \right)'$$

for all $z \in X$ and for some $\alpha \in L$, then $q(x, y) = 0^\sim$ implies that $\text{int}_{\tau_q} \mu(y) = \alpha \wedge 1 = \alpha$ for all $y \in X$ and all $\mu \in L^X$. Hence,

$$\alpha = \text{int}_{\tau_q} \mu(x) \leq \mu(x) < \beta \leq \text{int}_{\tau_q} \mu(y) = \alpha,$$

that is, $\alpha < \beta \leq \alpha$ which is a contradiction, and thus $x = y$ and q is an L -metric.

Now let $x \neq y$ and so $q(x, y) \neq 0^\sim$, then there exists $\alpha > 0$ such that $q(x, y)(\alpha) > 0$ and hence taking $v = \bar{1} \wedge R^\delta |_{\mathbf{R}_L^*} \circ q_x \in L^X$, we get that

$$v(y) = 1 \wedge R^\delta (q(x, y)) = 1 \wedge \left(\bigvee_{\eta \geq \delta} q(x, y)(\eta) \right)' < 1$$

whenever δ is chosen to be a very small number tends to zero. But $\text{int}_{\tau_q} v(x) = 1 \wedge \left(\bigvee_{\eta \geq \delta} q(x, x)(\eta) \right)' = 1$. Hence, (X, τ_q) is a GT_0 -space. $\square$

4. On L -uniform spaces

An L -filter $\mathcal{U}$ on $X \times X$ is called L -uniform structure on X [5] if the following conditions are fulfilled:

$$(U1) \ (x, x)^{\cdot} \succ \mathcal{U} \text{ for all } x \in X;$$

$$(U2) \ \mathcal{U} = \mathcal{U}^{-1};$$

$$(U3) \ \mathcal{U} \circ \mathcal{U} \succ \mathcal{U}.$$

Where $(x, x)^{\cdot}(u) = u(x, x)$, $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ and $(\mathcal{U} \circ \mathcal{U})(u) = \bigvee_{v \circ w \leq u} (\mathcal{U}(w) \wedge \mathcal{V}(v))$ for all $u \in L^{X \times X}$, and $u^{-1}(x, y) = u(y, x)$ and $(v \circ w)(x, y) = \bigvee_{z \in X} (w(x, z) \wedge v(z, y))$ for all $x, y \in X$.

A set X equipped with an L -uniform structure $\mathcal{U}$ is called an L -uniform space. A mapping $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ between L -uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is said to be L -uniformly continuous (or $(\mathcal{U}, \mathcal{V})$ -continuous) provided

$$\mathcal{F}_L(f \times f)(\mathcal{U}) \succ \mathcal{V}$$

holds. To each L -uniform structure $\mathcal{U}$ on X is associated a stratified L -topology $\tau_{\mathcal{U}}$. The related interior operator $\text{int}_{\mathcal{U}}$ is given by:

$$(\text{int}_{\mathcal{U}}\lambda)(x) = \mathcal{U}[\dot{x}](\lambda)$$

for all $x \in X$ and all $\lambda \in L^X$, where $\mathcal{U}[\dot{x}](\lambda) = \bigvee_{u[\mu] \leq \lambda} (\mathcal{U}(u) \wedge \mu(x))$ and $u[\mu](x) = \bigvee_{y \in X} (\mu(y) \wedge u(y, x))$. For all $x \in X$ and all $\lambda \in L^X$ we have

$$\mathcal{U}[\dot{x}] = \mathcal{N}(x) \quad \text{and} \quad \mathcal{U}[\dot{\lambda}] = \mathcal{N}(\lambda),$$

where $\mathcal{N}(x)$ and $\mathcal{N}(\lambda)$ are the L -neighborhood filters of the space $(X, \tau_{\mathcal{U}})$ at x and λ , respectively.

Let $\mathcal{U}$ be an L -uniform structure on a set X . Then $u \in L^{X \times X}$ is called a *surrounding* provided $\mathcal{U}(u) \geq \alpha$ for some $\alpha \in L_0$ and $u = u^{-1}$. A surrounding $u \in L^{X \times X}$ is called *left (right) invariant* provided

$$u(ax, ay) = u(x, y)(u(xa, ya) = u(x, y)) \text{ for all } a, x, y \in X.$$

$\mathcal{U}$ is called a *left (right) invariant L -uniform structure* if $\mathcal{U}$ has a valued L -filter base consists of left (right) invariant surroundings [4].

L -topological groups. In the following we focus our study on a multiplicative group G . We denote, as usual, the identity element of G by e and the inverse of an element a of G by a^{-1} . Let G be a group and τ an L -topology on G . Then (G, τ) will be called an L -topological group [2] if the mappings

$$\pi: (G \times G, \tau \times \tau) \rightarrow (G, \tau) \text{ defined by } \pi(a, b) = ab \text{ for all } a, b \in G$$

and

$$i: (G, \tau) \rightarrow (G, \tau) \text{ defined by } i(a) = a^{-1} \text{ for all } a \in G$$

are L -continuous. π and i are the binary operation and the unary operation of the inverse on G , respectively.

For all $\lambda \in L^G$, the inverse λ^i of λ with respect to the unary operation i on G is the L -set $\lambda \circ i$ in G defined by [4]

$$\lambda^i(x) = \lambda(x^{-1}) \text{ for all } x \in G.$$

Example 4.1. For a group G , the induced L -topological space $(G, \omega_L(T))$ of the usual topological group (G, T) is an L -topological group.

Proposition 4.1. [4] Let (G, τ) be an L -topological group. Then there exist on G a unique left invariant L -uniform structure $\mathcal{U}^l$ and a unique right invariant L -uniform structure $\mathcal{U}^r$ compatible with τ , constructed using the family $(\alpha - \text{pr } \mathcal{N}(e))_{\alpha \in L_0}$ of all filters $\alpha - \text{pr } \mathcal{N}(e)$, where $\mathcal{N}(e)$ is the L -neighborhood filter at the identity element e of (G, τ) , as follows:

$$\mathcal{U}^l(u) = \bigvee_{v \in \mathcal{U}_x^l, v \leq u} \alpha \quad \text{and} \quad \mathcal{U}^r(u) = \bigvee_{v \in \mathcal{U}_x^r, v \leq u} \alpha, \quad (4.1)$$

where

$$\mathcal{U}_x^l = \alpha - \text{pr } \mathcal{U}^l \quad \text{and} \quad \mathcal{U}_x^r = \alpha - \text{pr } \mathcal{U}^r \quad (4.2)$$

are defined by

$$\mathcal{U}_x^l = \{u \in L^{G \times G} \mid u(x, y) = (\lambda \wedge \lambda^i)(x^{-1}y) \text{ for some } \lambda \in \alpha - \text{pr } \mathcal{N}(e)\} \quad (4.3)$$

and

$$\mathcal{U}_x^r = \{u \in L^{G \times G} \mid u(x, y) = (\lambda \wedge \lambda^i)(xy^{-1}) \text{ for some } \lambda \in \alpha - \text{pr } \mathcal{N}(e)\} \quad (4.4)$$

We should notice that we shall fix the notations $\mathcal{U}^l, \mathcal{U}^r, \mathcal{U}_x^l$ and $\mathcal{U}_x^r$ along the paper to be these defined above.

Remark 4.1. For the L -topological group (G, τ) , the elements u of $\mathcal{U}_x^l(\mathcal{U}_x^r)$ are left (right) invariant surroundings. Moreover, $(\mathcal{U}_x^l)_{\alpha \in L_0}((\mathcal{U}_x^r)_{\alpha \in L_0})$ is a valued L -filter base for the left (right) invariant L -uniform structure $\mathcal{U}^l(\mathcal{U}^r)$ defined by (4.1), (4.2), (4.3), (4.4), respectively.

L -topogenous orders. A binary relation on L^X is said to be an L -topogenous order on X [13] if the following conditions are fulfilled:

- (1) $\bar{0} \ll \bar{0}$ and $\bar{1} \ll \bar{1}$;
- (2) $\lambda \mu$ implies $\lambda \leq \mu$;
- (3) $\lambda_1 \leq \lambda \mu \leq \mu_1$ implies $\lambda_1 \mu_1$;
- (4) From $\lambda_1 \mu_1$ and $\lambda_2 \mu_2$ it follows $\lambda_1 \vee \lambda_2 \mu_1 \vee \mu_2$ and $\lambda_1 \wedge \lambda_2 \mu_1 \wedge \mu_2$.

An L -topogenous order is said to be *regular* or is said to be an L -topogenous structure if for all $\lambda, \mu \in L^X$ with $\lambda \mu$ there is a $v \in L^X$ such that λv and $v \mu$ hold, and is called *complementarily symmetric* if $\lambda \mu$ implies $\mu' \lambda'$ for all $\lambda, \mu \in L^X$ and moreover is called *perfect* if for each family $(\lambda_i)_{i \in I}$ of L -subsets of X with $\lambda_i \mu$ for all $i \in I$ it follows $\bigvee_{i \in I} \lambda_i \ll \mu$.

Let $(\cdot_n)$ be a sequence of L -topogenous structures on X and $(\cdot_n)$ a sequence of L -topogenous structures on I_L . Then an L -real function $f: X \rightarrow I_L$ is said to be *associated with the sequence* $(\cdot_n)$ if for all $\lambda, \mu \in L^L$, $\lambda \cdot_n \mu$ implies $(\lambda \circ f)_{n+1}(\mu \circ f)$ for every positive integer n [11].

Now, suppose that (G, τ) has a countable L -neighborhood filter $\mathcal{N}(e)$ at the identity e . Since any L -topological group, from Proposition 4.1, is uniformizable, then the left and the right invariant L -uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$, constructed also in Proposition 4.1, has, from Remark 4.1, a countable L -filter base $\mathcal{U}_{\frac{1}{n}}^l$ and $\mathcal{U}_{\frac{1}{n}}^r$, respectively, $n \in \mathbb{N}$.

Lemma 4.1. [18] For all $\lambda, \mu \in L^X$, we have $\lambda \leq \mu$ if and only if $\dot{\lambda} \succ \dot{\mu}$.

Here, we prove this interesting result.

Lemma 4.2. Let $\mathcal{U}$ be an L -uniform structure on a set X , and define a binary relation on L^X as follows:

$$\lambda \ll_{\mathcal{U}} \mu \iff \mathcal{U}[\dot{\lambda}] \succ \dot{\mu}$$

for all $\lambda, \mu \in L^X$. Then $\ll_{\mathcal{U}}$ is a complementarily symmetric perfect L -topogenous order on X .

Proof. From the properties of $\mathcal{U}$ as an L -filter, (2.1) and Lemma 4.1 we get easily that $\ll_{\mathcal{U}}$ fulfills all the required conditions. $\square$

Proposition 4.2. [13] There is a one-to-one correspondence between the perfect L -topogenous structures on a set X and the L -topologies τ on X . This correspondence is given by

$$\lambda \ll \mu \iff \lambda \leq v \leq \mu \text{ for some } v \in \tau$$

for all $\lambda, \mu \in L^X$ and

$$\tau = \{\lambda \in L^X \mid \lambda \ll \lambda\}.$$

Now we have the following result.

Proposition 4.3. Suppose that $\mathcal{U}$ and $(\mathcal{U}_n)_{n \in \mathbb{N}}$ are an L -uniform structure on X and its countable L -filter base, respectively, and also consider $\mathcal{V}$ an L -uniform structure on I_L . Let $(\sigma_n)_{n \in \mathbb{N}}$ denote a sequence of complementarily symmetric perfect L -topogenous structures on X for which $\lambda \ll_{\sigma_n} \mu \iff \mathcal{U}[\dot{\lambda}] \succ \dot{\mu}$ for all $\lambda, \mu \in L^X$, and let Φ be the family of all L -uniformly continuous functions $h : (X, \mathcal{U}) \rightarrow (I_L, \mathcal{V})$ associated with $(\sigma_n)_{n \in \mathbb{N}}$. Then the mapping $\sigma_{\mathcal{U}} : X \times X \rightarrow I_L$ defined by

$$\sigma_{\mathcal{U}}(x, y) = \sup \{ \sigma_f(x, y) \mid f \in \Phi \},$$

where $\sigma_f(x, y) = (f(x) - f(y))^+$ for all $x, y \in X$, is an L -pseudo-metric on X and $\tau_{\mathcal{U}} = \tau_{\sigma_{\mathcal{U}}}$.

Proof. The proof of that $\sigma_{\mathcal{U}}$ is an L -pseudo-metric on X comes from Lemmas 3.1, 3.2, and 4.2.

Since for any $\lambda \in L^X$, and from Proposition 4.2

$$\lambda \ll_n \lambda \iff \mathcal{U}[\dot{\lambda}] \succ \dot{\lambda}$$

means that $\lambda \in \tau_{\mathcal{U}}$ if and only if $\lambda \in \tau_{\sigma_{\mathcal{U}}}$, where $\sigma_{\mathcal{U}}$ is generated by all the L -pseudo-metrics σ_h for every h associated with σ_n . Hence, $\tau_{\mathcal{U}} = \tau_{\sigma_{\mathcal{U}}}$. $\square$

5. The metrizable of L -topological groups

This section is devoted to show that any (separated) L -topological group is pseudo-metrizable (metrizable).

An L -topological group (G, τ) is called *separated* if for the identity element e , we have $\bigwedge_{\lambda \in \alpha - \text{pr} \mathcal{N}(e)} \lambda(e) \geq \alpha$, and $\bigwedge_{\lambda \in \alpha - \text{pr} \mathcal{N}(e)} \lambda(x) < \alpha$ for all $x \in G$ with $x \neq e$ and for all $\alpha \in L_0$ [4].

Proposition 5.1. [4] Any (separated) L -topological group is a $(GT_{3\frac{1}{2}}\text{-space})$ completely regular space.

Now, we are going to show the main result in this paper.

Proposition 5.2. Any (separated) L -topological group (G, τ) is pseudo-metrizable (metrizable).

Proof. From Proposition 4.1, we have unique left and unique right L -uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ on G defined by (4.1) such that $\tau = \tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r}$. Proposition 4.3 implies that $\tau = \tau_{\mathcal{U}^l} = \tau_{\sigma_{\mathcal{U}^l}}$ and $\tau = \tau_{\mathcal{U}^r} = \tau_{\sigma_{\mathcal{U}^r}}$, and therefore (G, τ) is pseudo-metrizable.

Also, if (G, τ) is separated, then from Proposition 5.1, we get that (G, τ) is a GT_0 -space, and hence, from Lemma 3.3, we have that (G, τ) is metrizable. $\square$

We also have the following important result.

Proposition 5.3. Let (G, τ) be a (separated) L -topological group. Then the following statements are equivalent.

- (1) τ is pseudo-metrizable (metrizable);
- (2) e has a countable L -neighborhood filter $\mathcal{N}(e)$;
- (3) τ can be induced by a left invariant L -pseudo-metric (L -metric);
- (4) τ can be induced by a right invariant L -pseudo-metric (L -metric).

Proof. (1) $\Rightarrow$ (2): Follows from Proposition 3.1

(2) $\Rightarrow$ (3): Let e has a countable L -neighborhood filter $\mathcal{N}(e)$, and let $\mathcal{U}^l_n$ be a countable L -filter base of the left invariant L -uniform structure $\mathcal{U}^l$, defined by (4.1), which is compatible with τ . Then, from Lemma 4.2, $\lambda \ll_{\mathcal{U}^l_n} \mu \iff \mathcal{U}^l_n[\dot{\lambda}] \succ \dot{\mu}$ for all $\lambda, \mu \in L^G$ defines a sequence of complementarily symmetric perfect L -topogenous structures on G . Taking $\mathcal{V}$ as an L -uniform structure on I_L and Φ as the family of all L -uniformly continuous functions $h : (G, \mathcal{U}^l) \rightarrow (I_L, \mathcal{V})$ associated with $\ll_{\mathcal{U}^l_n}$, we get, from Proposition 4.3, that the L -mapping $\sigma : G \times G \rightarrow I_L$ defined by $\sigma(x, y) = \sup \{ (f(x) - f(y))^+ \mid f \in \Phi \}$ is an L -pseudo-metric on G and $\tau = \tau_{\mathcal{U}^l} = \tau_{\sigma_{\mathcal{U}^l}}$.

Now, we define $h_a : G \rightarrow I_L$ by $h_a(x) = h(ax)$ for all $a, x \in G$. From $h \in \Phi$ is L -uniformly continuous, that is, $\mathcal{F}_L(h \times h)(\mathcal{U}^l) \succ \mathcal{V}$ and that the elements of $\mathcal{U}^l_n$ are left invariant from Remark 4.1, and from (4.1), we have

$$\begin{aligned} \mathcal{F}_L(h_a \times h_a)(\mathcal{U}^l)(v) &= \mathcal{U}^l(v \circ (h_a \times h_a)) \\ &= \bigvee_{u \in \mathcal{U}^l_n, u \leq v \circ (h_a \times h_a)} \frac{1}{n} \\ &= \bigvee_{u \in \mathcal{U}^l_n, u \leq v \circ (h \times h)} \frac{1}{n} \\ &= \mathcal{F}_L(h \times h)(\mathcal{U}^l)(v) \\ &\geq \mathcal{V}(v). \end{aligned}$$

Hence, h_a is L -uniformly continuous associated with $\ll_{\mathcal{U}^l}$, that is, $h_a \in \Phi$. Thus

$$\begin{aligned} \sigma(ax, ay) &= \sup \{ (h(ax) - h(ay))^+ \mid h \in \Phi \} \\ &= \sup \{ (h_a(x) - h_a(y))^+ \mid h \in \Phi \} \\ &\leq \sup \{ (k(x) - k(y))^+ \mid k \in \Phi \} \\ &= \sigma(x, y). \end{aligned}$$

Applying the same for a^{-1} instead of a , we get that $\sigma(x, y) = \sigma(a^{-1}a x, a^{-1}a y) \leq \sigma(a x, a y)$. That is, $\sigma(a x, a y) = \sigma(x, y)$ for all $a, x, y \in G$ and then σ is a left invariant L -pseudo-metric on G inducing τ .

(2) $\Rightarrow$ (4): By a similar proof as in the case (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1): Obvious.

The proposition is still true if we consider the parentheses. $\square$

Example 5.1. From Proposition 5.2, we can deduce that any L -topological group (G, τ) on which there can be constructed an L -uniform structure $\mathcal{U}$ compatible with τ is pseudo-metrizable in general and is metrizable whenever (G, τ) is separated.

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